



1.6.8.8 Integer powers of complex numbers

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Introduction

- In this topic, we will
 - Define integer powers of a complex number
 - Derive a formula from the polar representation
 - Look at a geometric interpretation
 - Observe some properties





Integer exponents

- Given $z = \alpha + \beta j = |z| \angle \theta$, then for an integer n , we define

$$z^n = \begin{cases} (z^{-1})^{-n} & n < 0 \\ 1 & n = 0 \\ zz^{n-1} & n > 0 \end{cases}$$

- For example, $(1.6 + 1.2j)^{-10}$ would be calculated by using $(0.4 - 0.3j)^{10}$
- Note that $0^0 = 1$, $0^n = 0$ if $n > 0$, and 0^n is not defined if $n < 0$





Polar representation

- Now, if $z = |z| \angle \theta$, observe that

$$\begin{aligned} z^2 &= z \cdot z \\ &= (|z| \angle \theta)(|z| \angle \theta) \\ &= |z|^2 \angle (2\theta) \end{aligned}$$

$$\begin{aligned} z^3 &= z \cdot z^2 \\ &= (|z| \angle \theta)(|z|^2 \angle (2\theta)) \\ &= |z|^3 \angle (3\theta) \end{aligned}$$





Polar representation

- Similarly, if $z = |z| \angle \theta$, $z^{-1} = |z|^{-1} \angle (-\theta)$ so

$$z^{-2} = z^{-1} \cdot z^{-1}$$

$$= \left(|z|^{-1} \angle (-\theta) \right) \left(|z|^{-1} \angle (-\theta) \right)$$

$$= |z|^{-2} \angle (-2\theta)$$

$$z^{-3} = z^{-1} \cdot z^{-2}$$

$$= \left(|z|^{-1} \angle (-\theta) \right) \left(|z|^{-2} \angle (-2\theta) \right)$$

$$= |z|^{-3} \angle (-3\theta)$$





Polar representation

- This suggests $z^n = |z|^n \angle(n\theta)$ for all integer values of n
 - We could even use this to define

$$z^\alpha = |z|^\alpha \angle(\alpha\theta)$$

for any real α

- In this course, we will restrict ourselves to polynomials that require integer exponents





Complex division

- As an example: $z = 1 + j = \sqrt{2} \angle \frac{\pi}{4}$

$$z^2 = (1 + j)^2 = \sqrt{2}^2 \angle \left(2 \frac{\pi}{4} \right) = 2j$$

$$z^3 = (1 + j)^3 = \sqrt{2}^3 \angle \left(3 \frac{\pi}{4} \right) = -2 + 2j$$

$$z^4 = (1 + j)^4 = \sqrt{2}^4 \angle \left(4 \frac{\pi}{4} \right) = -4$$

$$z^5 = (1 + j)^5 = \sqrt{2}^5 \angle \left(5 \frac{\pi}{4} \right) = -4 - 4j$$





Complex division

- As an example: $z = 1 + j = \sqrt{2} \angle \frac{\pi}{4}$ so $z^{-1} = \frac{1}{2} - \frac{1}{2}j = \frac{1}{\sqrt{2}} \angle \left(-\frac{\pi}{4}\right)$

$$z^{-2} = (1 + j)^{-2} = \sqrt{2}^{-2} \angle \left(-2\frac{\pi}{4}\right) = -\frac{1}{2}j$$

$$z^{-3} = (1 + j)^{-3} = \sqrt{2}^{-3} \angle \left(-3\frac{\pi}{4}\right) = -\frac{1}{4} - \frac{1}{4}j$$

$$z^{-4} = (1 + j)^{-4} = \sqrt{2}^{-4} \angle \left(-4\frac{\pi}{4}\right) = -\frac{1}{4}$$

$$z^{-5} = (1 + j)^{-5} = \sqrt{2}^{-5} \angle \left(-5\frac{\pi}{4}\right) = -\frac{1}{8} + \frac{1}{8}j$$





Complex division

- Note that z^n and z^{-n} are multiplicative inverses of each other:

$$z^2 = 2j$$

$$z^{-2} = -\frac{1}{2}j$$

$$z^3 = -2 + 2j$$

$$z^{-3} = -\frac{1}{4} - \frac{1}{4}j$$

$$z^4 = -4$$

$$z^{-4} = -\frac{1}{4}$$

$$z^5 = -4 - 4j$$

$$z^{-5} = -\frac{1}{8} + \frac{1}{8}j$$





Proof of formula

$$z^n = \begin{cases} (z^{-1})^{-n} & n < 0 \\ 0 & n = 0 \\ zz^{n-1} & n > 0 \end{cases}$$

Theorem: $z^n = |z|^n \angle (n\theta)$

Proof: When $n = 0$, $z^0 = |z|^0 \angle (0 \cdot \theta) = 1 \angle 0 = 1$

Assume the formula is correct for an arbitrary $n \geq 0$,
that is, $z^n = |z|^n \angle (n\theta)$.

Then, $z^{n+1} = z \cdot z^n$

$$= (|z| \angle \theta) (|z|^n \angle (n\theta))$$

$$= |z| |z|^n \angle (\theta + n\theta)$$

$$= |z|^{n+1} \angle ((n+1)\theta)$$

Thus, by the process of mathematical induction, the statement is true for all $n = 0, 1, 2, 3, \dots$.





Proof of formula

$$z^n = \begin{cases} (z^{-1})^{-n} & n < 0 \\ 0 & n = 0 \\ zz^{n-1} & n > 0 \end{cases}$$

Theorem: $z^n = |z|^n \angle(n\theta)$

Proof: When $-n < 0$, the polar representation is unchanged:

$$z^{-n} = |z|^{-n} \angle(-n\theta)$$

The formula, however, says for $-n < 0$ that

$$\begin{aligned} z^{-n} &= (z^{-1})^n \\ &= (|z|^{-1} \angle(-\theta))^n \\ &= (|z|^{-1})^n \angle(n(-\theta)) \\ &= |z|^{-n} \angle(-n\theta) \end{aligned}$$

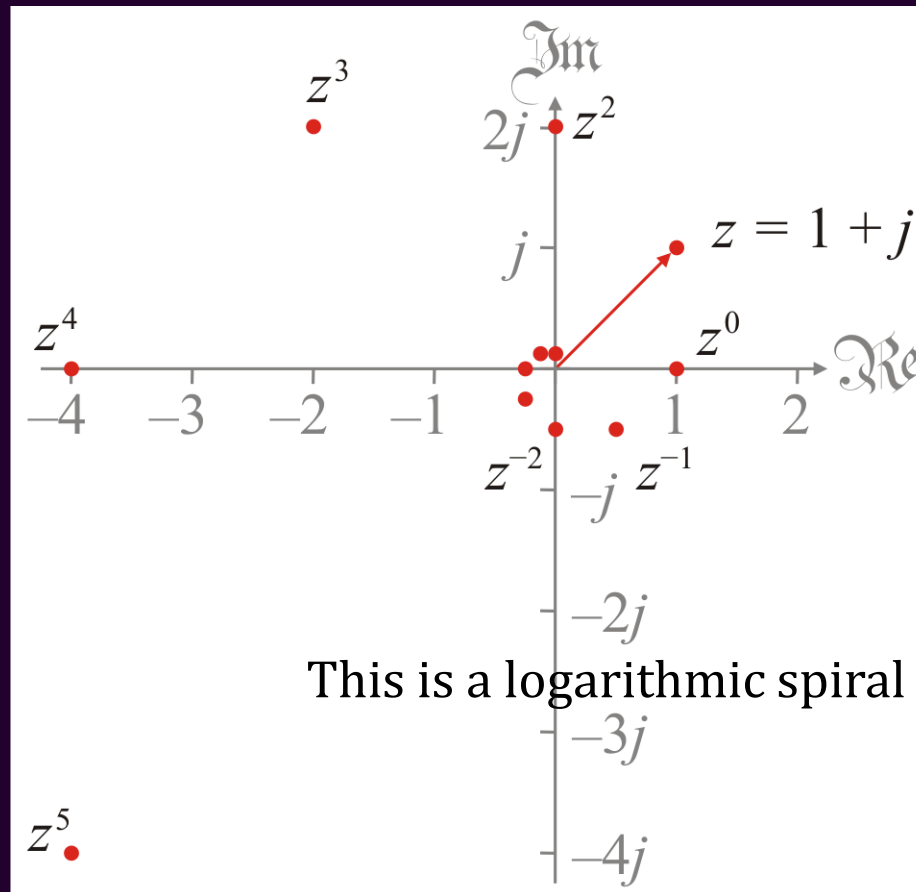
Thus, the formula is correct for all n . $\mathcal{QED}$





Geometric interpretation

- The geometric interpretation of integer powers is a spiral
 - Powers of $z = 1 + j$



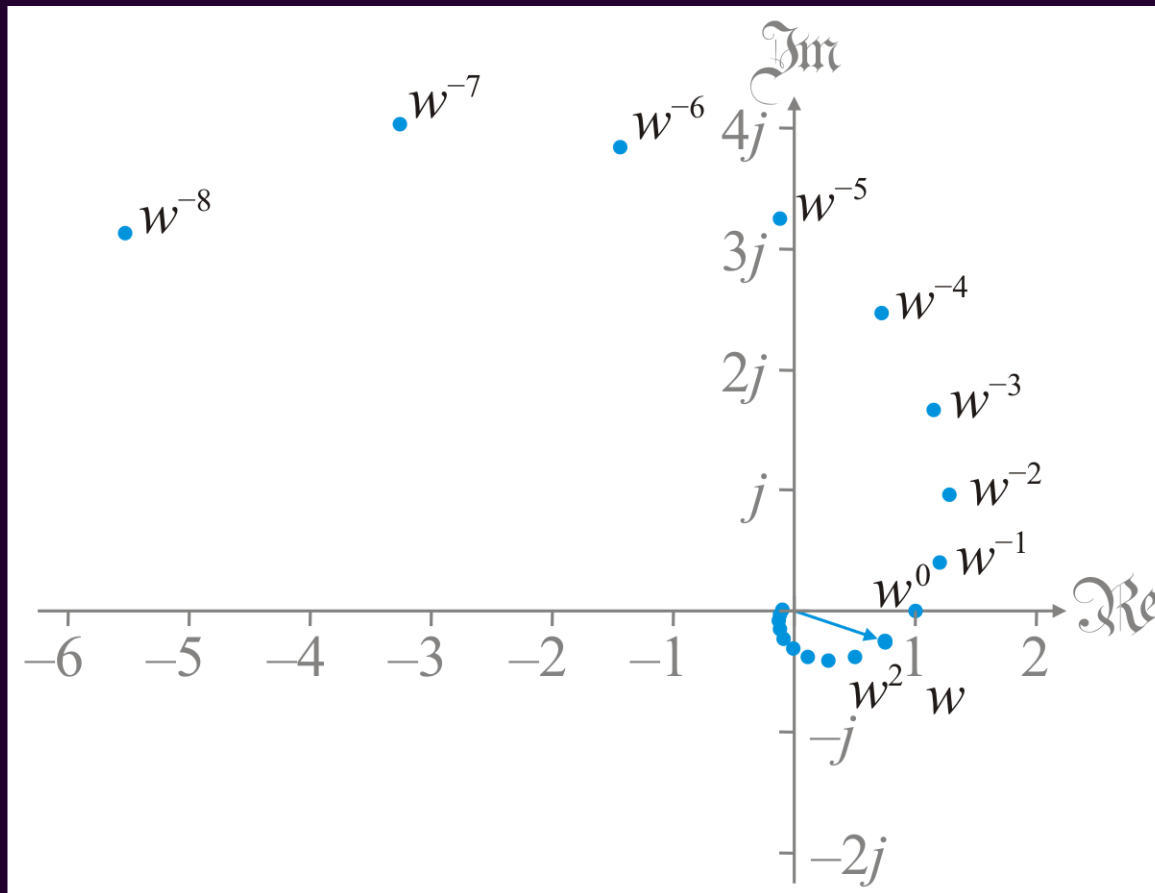
$$|z| = \sqrt{2} > 1$$





Geometric interpretation

- The geometric interpretation of integer powers is a spiral
 - Powers of $w = 0.75 - 0.25j$ where $w^{-1} = 1.2 + 0.4j$



$$|w| = \frac{\sqrt{10}}{4} < 1$$

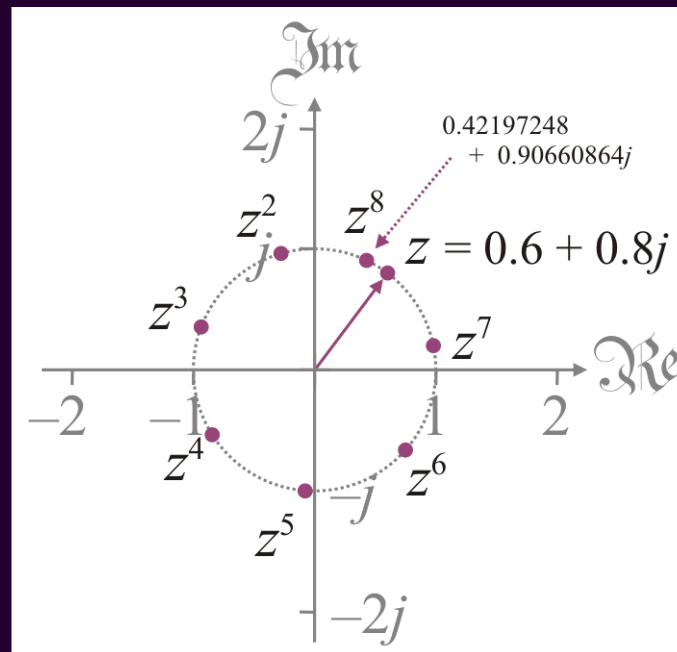




Geometric interpretation

- The geometric interpretation of integer powers is a spiral
 - If $|z| = 1$, then all powers remain on the unit circle
 - Here $z = 0.6 + 0.8j \approx 1 \angle 53.13^\circ$

$$z^n \approx 1 \angle (n53.13^\circ)$$





Properties

Theorem: $|z^n| = |z|^n$

Proof: Obvious from the polar representation. *QED*





Properties

Theorem: If $z = \alpha + 0j$ is a positive real number then $z^n = \alpha^n$.

Proof: If $z = \alpha + 0j$ where $\alpha > 0$, then $|z| = |\alpha| = \alpha$, so $z = \alpha \angle 0$,

$$\begin{aligned} z^n &= \alpha^n \angle (n \cdot 0) \\ &= \alpha^n \angle 0 = \alpha^n \quad \text{QED} \end{aligned}$$





Properties

Theorem: If $z = -\alpha + 0j$ is a negative real number then $z^n = (-\alpha)^n$.

Proof: If $z = -\alpha + 0j$ where $\alpha > 0$, then $|z| = |\alpha| = \alpha$, so $z = \alpha \angle \pi$

$$\begin{aligned} z^n &= \alpha^n \angle (n\pi) \\ &= \alpha^n (\cos(n\pi) + \sin(n\pi)j) \\ &= \alpha^n (-1)^n \\ &= (\alpha(-1))^n \\ &= (-\alpha)^n \quad \text{Q.E.D.} \end{aligned}$$





Properties

Theorem: $z^m z^n = z^{m+n}$

Proof: This is also straight-forward:

$$\begin{aligned} z^m z^n &= \left(|z|^m \angle (m\theta) \right) \left(|z|^n \angle (n\theta) \right) \\ &= |z|^m |z|^n \angle (m\theta + n\theta) \\ &= |z|^{m+n} \angle ((m+n)\theta) = z^{m+n} \quad \text{QED} \end{aligned}$$





Summary

- In this topic, we've introduced integer powers of complex numbers
 - We used the same definition as for integer powers of real numbers
 - We derived the polar representation
 - We saw the geometric interpretation
 - Rather beautiful spirals
 - We saw a number of properties





References

- [1] https://en.wikipedia.org/wiki/Complex_number#Exponentiation
- [2] https://en.wikipedia.org/wiki/Logarithmic_spiral





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

<https://www.rbg.ca/>

for more information.





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